1202 Summer 2010: Solutions

1 H is a subgroup of G if (A) $e \in H$, (B) $g, h \in H \Rightarrow gh \in H$ and (C) $g \in H \Rightarrow g^{-1} \in H$.

(i) (i) $G = GL_2(\mathbb{R})$, the group of 2×2 invertible real matrices under multiplication, $H = \{A \in G : A = A^T\}$. Clearly the identity of G, which is I_2, is an element of H. However, H is not closed under the group operation, for example, let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. Clearly A and B are in H, but $AB = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \notin H$. Thus H is not a subgroup.

(ii) $G = GL_2(\mathbb{R})$, $H = \{\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in \mathbb{R}, \alpha \geq 0\}$. Again the identity is clearly in H, but in this case H is not closed under inverses: for example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in H$, but $A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \notin H$. Thus H is not a subgroup

(iii) $G = GL_2(\mathbb{R})$, $H = \{\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} : \beta \in \mathbb{Z}\}$. The identity I_2 is in H. Let $A, B \in H$, say $A = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, where $\beta, \gamma \in \mathbb{Z}$. Then $AB = \begin{pmatrix} 1 & 0 \\ \beta + \gamma & 1 \end{pmatrix} \in H$, since $\beta + \gamma \in \mathbb{Z}$. Also $A^{-1} = \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \in H$. Thus H is a subgroup.

(iv) G is any abelian group, $H = \{g \in G : g^2 = e\}$. Since $e^2 = e$, $e \in H$. Let $g, h \in H$, so $g^2 = h^2 = e$. Then, since G is abelian, $(gh)^2 = ghgh = g^2h^2 = ee = e$, and hence $gh \in H$. Also $g^{-1} = g \in H$. Thus H is a subgroup

(v) $G = S_4$, $H = \{g \in G : g^2 = e\}$. Let $g = (1 \; 2)$, $h = (2 \; 3)$. Then clearly $g^2 = h^2 = e$, so $g, h \in H$. However, $gh = (1 \; 2 \; 3) \notin H$. Thus H is not a subgroup.

2. (a) Let G be a finite group and H a subgroup. Then $|H|$ divides $|G|$.

The *order* of an element g is the least $i > 0$ such that $g^i = e$.

Let G be a finite group and $g \in G$. The set $\{g^i : i \in \mathbb{Z}\}$ forms a subgroup of G and is of order $o(g)$. Hence by Lagrange's Theorem $o(g) = |H|$ divides $|G|$.

(b) $\mathbb{Z}_p^{\bullet}$ is of order $p - 1$, so by (a), we have that $o(\bar{a})$ divides $p - 1$, say $p - a = ro(\bar{a})$. $\bar{a}^{p-1} = \bar{a}^{o(\bar{a})r} = \bar{1}^r = \bar{1}$, as required.

(c) In $\mathbb{Z}_{17}^{\bullet}$, $\bar{2}^{16} = \bar{1}$ by (b). Hence $\bar{2}^{1605} = \bar{2}^{16\times100+5} = \bar{2}^5 = \overline{32} = \overline{15}$

$\bar{2}^{-1} = \bar{9}$, so $\bar{2}^{798} = \bar{2}^{800-2} = \bar{2}^{16\times50}\bar{2}^{-2} = \bar{9}^2 = \overline{81} = \overline{13}$.

(d) Suppose $\bar{x}^{11} = \bar{7}$. Then $(\bar{x}^{11})^3 = \bar{7}^3 = \overline{49} \times \bar{7} = \overline{15} \times \bar{7} = \overline{105} = \bar{3}$. But $(\bar{x}^{11})^3 = \bar{x}^{33} = \bar{x}^{2\times16+1} = \bar{x}$.
Thus $\bar{x} = \bar{3}$ is the solution.

3. (a) $\det A = \sum_{\sigma \in S_n} sign(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{n,\sigma(n)}$

Let B be obtained from A by performing the elementary row operation e. Then if

(a) $e = p(r,s)$ (exchange rows r and s), then $\det B = -\det A$;

(b) $e = d(r;\lambda)$ (multiply row r by λ), then $\det B = \lambda \det A$;

(c) $e = r(i,j;\lambda)$ (add λ times row i to row j) then $\det B = \det A$.

(b) $$\det \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 6 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
$$= \det \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -1.$$

(c) Either done like this:

Below we use the following:

(1) is take col 1 from col 2, col 3 and col 4; (2) is expand along first row

(3) is divide col 1 by $y-x$, col 2 by $z-x$, col 3 by $t-x$

(4) is take col 1 from col 2 and col 3; (5) is expand along first row

(6) is divide col 1 by $z-y$, col 2 by $t-y$

(7) is take col 1 from col 2; (8) is expand along first row.

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ x^3 & y^3 & z^3 & t^3 \end{pmatrix} \overset{(1)}{=} \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & y-x & z-x & t-x \\ x^2 & y^2-x^2 & z^2-x^2 & t^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 & t^3-x^3 \end{pmatrix}$$
$$\overset{(2)}{=} \det \begin{pmatrix} y-x & z-x & t-x \\ y^2-x^2 & z^2-x^2 & t^2-x^2 \\ y^3-x^3 & z^3-x^3 & t^3-x^3 \end{pmatrix}$$
$$\overset{(3)}{=} (y-x)(z-x)(t-x) \det \begin{pmatrix} 1 & 1 & 1 \\ y+x & z+x & t+x \\ y^2+xy+x^2 & z^2+xz+x^2 & t^2+xt+x^2 \end{pmatrix}$$

$\overset{(4)}{=} (y-x)(z-x)(t-x) \det \begin{pmatrix} 1 & 0 & 0 \\ y+x & z-y & t-y \\ y^2+xy+x^2 & z^2-y^2+xz-xy & t^2-y^2+xt-xz \end{pmatrix}$

$\overset{(5)}{=} (y-x)(z-x)(t-x) \det \begin{pmatrix} z-y & t-y \\ z^2-y^2+xz-xy & t^2-y^2+xt-xy \end{pmatrix}$

$\overset{(6)}{=} (y-x)(z-x)(t-x)(z-y)(t-y) \det \begin{pmatrix} 1 & 1 \\ x+y+z & x+y+t \end{pmatrix}$

$\overset{(7)}{=} (y-x)(z-x)(t-x)(z-y)(t-y) \det \begin{pmatrix} 1 & 0 \\ x+y+z & t-z \end{pmatrix}$

$\overset{(8)}{=} ((y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$

OR done like this:

Let $f(x,y,z,t) = \det A$. Note that if $x=y$ then two columns are the same and so $f(x,x,z,t)=0$. This means that $y-x$ divides $\det A$. Similarly $z-x$, $t-x$, $z-y$, $t-y$, $t-z$ divide f.
Hence $f = (y-x)(z-x)(t-x)(z-y)(t-y)(t-z)g(x,y,z,t)$.
But all terms in the determinant are clearly of total degree 6, so in fact g must be a constant. There is just one term yz^2t^3 in the determinant, which appears with a plus sign as it corresponds to the permutation $\sigma = id$. But the coefficient of yz^2t^3 in $(y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$ is also $+1$; hence $g=1$ and so $\det A = (y-x)(z-x)(t-x)(z-y)(t-y)(t-z)$

Since the given matrix is a particular example with $x=1, y=2, z=-2, t=-1$ and all these numbers are distinct, the determinant is non-zero and hence the matrix is invertible.

4. (i) $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$.

$c_A(t) = \det \begin{pmatrix} t-2 & -2 \\ -2 & t-5 \end{pmatrix} = t^2 - 7t + 6 = (t-1)(t-6)$. Hence eigenvalues of A are 1 and 6.

$\lambda = 1$; then eigenvector is solution to $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$, e.g. $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Similarly $\lambda = 6$ yields eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Hence if we take $P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, then $P^{-1}AP = D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ is diagonal.

(ii) Here $A^n = (PDP^{-1})^n = PD^nP^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6^n \end{pmatrix} (1/5) \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} =$
$(1/5) \begin{pmatrix} 2 & 6^n \\ -1 & 2(6^n) \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = (1/5) \begin{pmatrix} 4+6^n & -2+2(6^n) \\ -2+2(6^n) & 1+4(6^n) \end{pmatrix}$

(iii) $\begin{aligned} \frac{dx}{dt} &= 2x + 2y \\ \frac{dy}{dt} &= 2x + 5y \end{aligned}$. Write $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then we can write these equations as $\mathbf{x}' = A\mathbf{x}$, where the $'$ denotes differentiation with respect to t. Now make a change of variable $\mathbf{x} = P\mathbf{X}$. Then the equation becomes: $P\mathbf{X}' = AP\mathbf{X}$, so $\mathbf{X}' = (P^{-1}AP)\mathbf{X}$, i.e. $\mathbf{X}' = D\mathbf{X}$.

Writing this out, we have
$\begin{pmatrix} dX/dt \\ dY/dt \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ i.e. $\begin{aligned} \frac{dX}{dt} &= X \\ \frac{dY}{dt} &= 6Y \end{aligned}$

This has solutions $X = Ae^t$, $Y = Be^{6t}$. Hence $\mathbf{X} = \begin{pmatrix} Ae^t \\ Be^{6t} \end{pmatrix}$.

Now $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; it follows that

$\mathbf{X}(0) = P^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1/5) \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 2/5 \end{pmatrix}$

Hence $A = -1/5$, $B = 2/5$, and so $\mathbf{X} = (1/5) \begin{pmatrix} -e^t \\ 2e^{6t} \end{pmatrix}$. Thus

$\mathbf{x} = (1/5) \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} (1/5) \begin{pmatrix} -e^t \\ 2e^{6t} \end{pmatrix} = (1/5) \begin{pmatrix} -2e^t + 2e^{6t} \\ e^t + 4e^{6t} \end{pmatrix}$.

5. (a) (i) The *eigenspace* $E_{\lambda_i} = \{\mathbf{v} \in \mathbb{C}^n : A\mathbf{v} = \lambda_i \mathbf{v}\}$.
(ii) The *geometric multiplicity* e_i is $\dim(E_{\lambda_i})$.
(iii) The *characteristic polynomial* $c_A(t)$ of A is given by $\det(tI - A)$.
(iv) Write $c_A(t) = (t-\lambda_1)^{f_1} ... (t-\lambda_r)^{f_r}$. Then the *algebraic multiplicity* of A is f_i.

(b) Suppose $\sum_i^r \mathbf{u}_i = \mathbf{0}$ where not all $\mathbf{u}_i = \mathbf{0}$ ($\mathbf{u}_i \in E_{\lambda_i}$). Then pick a shortest non-trivial relation that holds; by renumbering, we can assume the relation is

$$\sum_{i=1}^p \mathbf{u}_i = 0 \qquad (1)$$

where each $\mathbf{u}_i \neq \mathbf{0}$.
Then $A \sum_{i=1}^p \mathbf{u}_i = \mathbf{0}$, so

$$\sum_{i=1}^p \lambda_i \mathbf{u}_i = 0 \qquad (2)$$

Now taking eqn (2) from λ_p times equation (1) we get

$$\sum_{i=1}^{p-1} (\lambda_p - \lambda_i)\mathbf{u}_i = 0 \qquad (3)$$

and this is a shorter relation than (1) (since it involves at most $p-1$ terms) and is also non-trivial (since the terms are non-zero since $\lambda_p \neq \lambda_i$). This is a contradiction.
So the only possibility is that there are no such relations, i.e. the sum is direct.

Hence if $\sum_i e_{i=1}^r = n$, then $dim(\oplus_{i=1}^r E_{\lambda_i}) = \sum_i^r e_i = n$ and hence $\oplus_{i=1}^r E_{\lambda_i} = \mathbb{C}^n$. Let $\mathcal{B}_i$ be a basis for E_{λ_i}. Then $\mathcal{B} = \cup_{i=1}^r \mathcal{B}_i$ is a basis for $\mathbb{C}^n$ consisting of eigenvectors: hence A is diagonalisable.

(c) A is diagonalisable if and only if $e_i = f_i$ $(i = 1, ..., r)$.

$A = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Quick calculation gives $c_A(t) = (t-3)(t-2)^3$.

Hence eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$ with $f_1 = 1$ and $f_2 = 3$.
Find e_2. $E_2 = \{\mathbf{v} \in \mathbb{C}^4 : A\mathbf{v} = 2\mathbf{v}\}$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Here there are 4 variables and 3 leading ones, so the dimension of E_2 is 1: thus $1 = e_2 < f_2 = 3$ and hence A is not diagonalisable.

6. (a) Apply the Gram-Schmidt process. $v_1(x) = 1$, $v_2(x) = x$, $v_3(x) = x^2$.
$|v_1|^2 = \int_0^1 1dx = 1$, so $f_1 = 1$.
$\langle v_2, f_1\rangle = \int_0^1 xdx = 1/2$. Hence we define
$w_2(x) = v_2(x) - \langle v_1, f_2\rangle f_1(x) = x - 1/2$.
$|w_2|^2 = \int_0^1 (x-1/2)^2 dx = 1/12$, so we define
$f_2(x) = \sqrt{12}(x - 1/2)$.
$w_3 = v_3 - \langle v_3, f_1\rangle f_1 - \langle v_3, f_2\rangle f_2$
$\langle v_3, f_1\rangle = \int_0^1 x^2 dx = 1/3$.
$\langle v_3, f_2\rangle = \int_0^1 \sqrt{12}x^2(x-1/2) = \sqrt{12}(1/4 - 1/6) = 1/\sqrt{12}$.
Hence $w_3 = x^2 - \frac{1}{3} - (x - 1/2) = x^2 - x + 1/6$.
$\|w_2\|^2 = \int_0^1 (x^2 - x + 1/6)^2 = \int_0^1 x^4 + x^2 + 1/36 - 2x^3 + 1/3x^2 - 1/3x = 1/5 + 1/3 + 1/36 - 1/2 + 1/9 - 1/6 = 1/180$
Hence $f_3 = \sqrt{180}(x^2 - x + 1/6)$.
So an orthonormal basis for W is given by $\{f_1, f_2, f_3\}$, where $f_1(x) = 1$, $f_2(x) = \sqrt{12}(x - 1/2)$ and $f_3(x) = \sqrt{180}(x^2 - x + 1/6)$.

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. $c_A(t) = (t-2)^2 - 1 = (t-1)(t-3)$. Hence eigenvalues are 1 and 3, and corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. These are already orthogonal, so we must just normalize them to get the orthogonal matrix
$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Then $P^{-1}AP = P^TAP = D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(c) $2x^2 + 2xy + y^2 = 1$. Write $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Equation can be written as $\mathbf{v}^TA\mathbf{v} = 1$. Let $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ and let $\mathbf{v} = P\mathbf{u}$. Then in new co-ordinates equation is $\mathbf{u}^TP^TAP\mathbf{u} = 1$, i.e. $\mathbf{u}^tD\mathbf{u} = 0$. Writing this in co-ordinates gives $u^2 + 3v^2 = 1$.
This is an ellipse and graph in (u, v)-plane shown in Diagram 1 below. To sketch this on (x, y)-plane, note that $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to $\begin{pmatrix} x \\ y \end{pmatrix} =$

$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponds to $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$; see Diagram 2.

Diagram 1

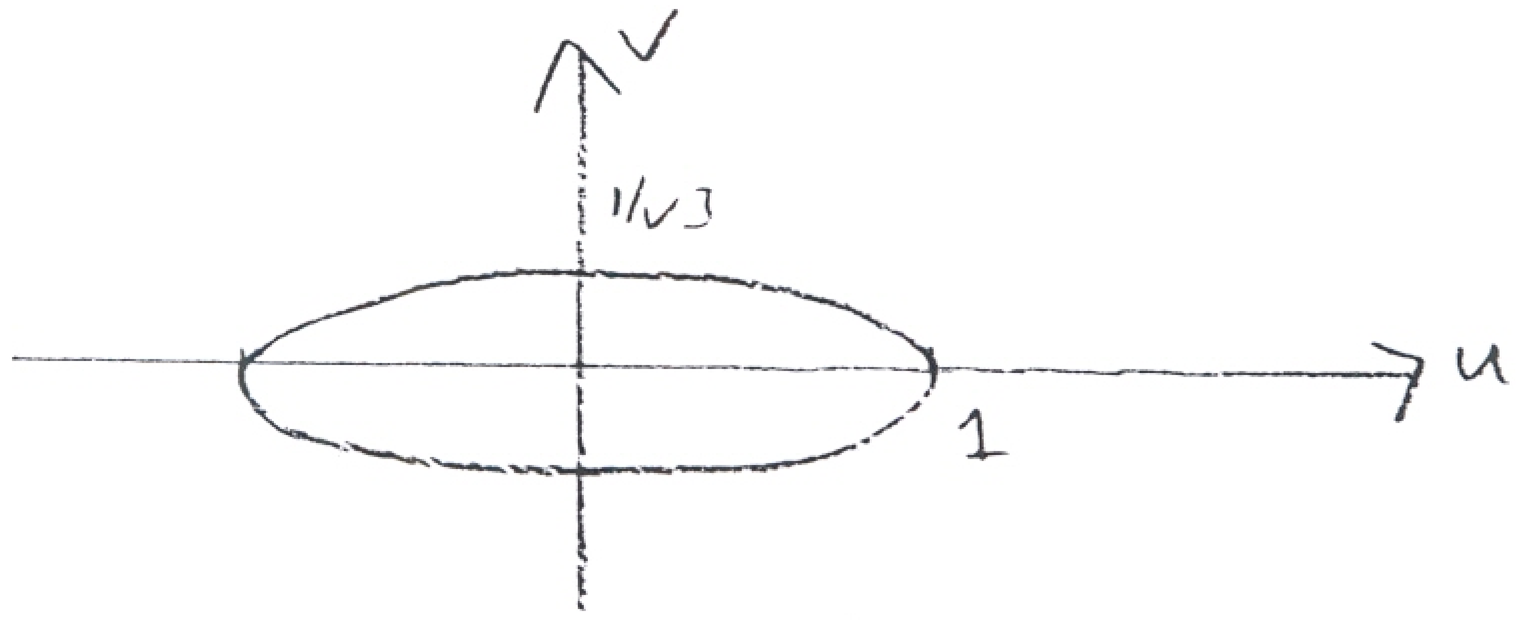

Diagram 2